Short Note on Fortuity

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Table Of Contents I

Introducion and Motivation

In AdS/CFT correspondence, we have the relation $g_s \sim g_{YM}^2$ while fixing $\lambda = g_{YM}^2 N$, this implies the weak string coupling when N goes large and strong coupling when N is finite. Indicating we have a classical bulk theory for $N \to \infty$, and quantum effects for finite N.

Since boundary holographic CFT is often calculable, we can analyze the behavior of SUSY protected observables (states) to know the behavior of the bulk gravity theory. This enables us to understand gravity theory with highly quantum behavior by studying BPS states at finite rank N.

Introducion and Motivation

We can define BPS states rigorously by supercharge action. Since we generally have

$$\{Q^I_\alpha, Q^J_\beta\} = 0 \tag{1}$$

in SUSY algebra, we can find a specific composition Q that satisfies $Q^2 = 0$. This Q acts on operators (we denote as $Q\mathcal{O}$ after)

$$[Q, \mathcal{O}\} = \cdots \tag{2}$$

We also can define a similar $Q^\dagger$, which gives

$$\{Q, Q^\dagger\} = (\mathit{some\ condition}) \tag{3}$$

whenever the state satisfies the RHS condition, many of its contributions are protected by the global SUSY symmetry, which is called a BPS state.

$$BPS \leftrightarrow Q\,|\psi\rangle = Q^\dagger\,|\psi\rangle = 0 \tag{4}$$

Thus, by Hodge decomposition, we can define BPS state by the cohomology of Q complex

$$H^p_{BPS} = \frac{\ker Q}{\mathrm{Im} Q} \tag{5}$$

where p denotes the different charge sector connected by Q acting.

However, BPS states can be classified further, because the matrix nature of gauge invariant BPS operators. Since some operators are not generally BPS, as

$$Q\mathcal{O} = (function\ of\ \ matrices) \tag{6}$$

and there are some trace relation for different matrix rank, these trace relations played a central role

$$2\mathrm{Tr}X^3 - 3\mathrm{Tr}X\mathrm{Tr}X^2 + (\mathrm{Tr}X)^3 = 0 \quad N = 2 \tag{7}$$

Therefore the when the function of matrices is a trace relation, we would have accidental BPS state , or **Fortuitous** state. And its opposite are named as **Monotone** state.

People have studied various models which possess a fortuity behavior, such like $4\mathrm{d}\ \mathcal{N}=4$ SYM[2, 3, 4], $D1-D5$ CFT[5, 6, 7], $\mathcal{N}=2$ SUSY SYK[8, 9], ABJM[10, 11, 12], ABJ, etc.

Introducion and Motivation

Why this interests? People suspect fortuitous operators corresponds to blackholes and fortuitous states are thus BH microstates. In contrast, monotone operators are gravitons.

This conjecture has a various of evidences. In [2], people find that some monotone states cannot dress the fortuitous states (which means, their product does not remains a nontrivial BPS class, in specific, become an exact form), this is a analogy to the BH no-hair theorem, named as **partial no-hair behavior**. Also, people believe that the growing of some fortuitous sector reproduces the growth of **blackhole entropy** [3].

Trace relations

We first take a glance of **trace relations**, generally, it's the identity relation

$$\sum_a c_a \prod_r \mathrm{Tr}(W_{a,r}) = 0 \tag{8}$$

The first question we can ask is how to generally obtain all possible trace relation for a fixed N, the fundamental principle is Cayley-Hamilton theorem, but we prefer to work on a fundamental relation for simplicity: For **any** $N \times N$ matrices $X_1, \cdots, X_{N+1}$, we have

$$F_N(X_1, \cdots, X_{N+1}) = \sum_{\sigma \in S_{N+1}} (-1)^\sigma \mathrm{Tr}_\sigma(X_1, \cdots, X_{N+1}) \tag{9}$$

where σ represents a permutation, for $\sigma = (i_1 \cdots i_r)(j_1 \cdots j_s) \cdots$, we define

$$\mathrm{Tr}_\sigma = \mathrm{Tr}(X_{i_1} \cdots X_{i_r})\mathrm{Tr}(X_{j_1} \cdots X_{j_s}) \cdots \tag{10}$$

Razmyslov and Procesi have proved this relation generates all possible trace relations in 1970s.

Trace relations

Second question we can ask is natural but a bit complicated, for any M matrices $X_1, \cdots, X_M$ in $N \times N$, how many independent trace relations we can find?

We should first fix the **multidegree** $(n_1, \ldots, n_M)$, which counts how many X_i's are their, we can get the **formal structures** by these multidegree. For example, $(n_X, n_Y) = (2, 1)$, our formal structures are

$$\mathrm{Tr}(X^2 Y), \mathrm{Tr}(X^2)\mathrm{Tr}(Y), \mathrm{Tr}(XY)\mathrm{Tr}(X), (\mathrm{Tr}X)^2\mathrm{Tr}\,Y \tag{11}$$

Trace relations

Razmyslov claims, the total number of independent trace relations, or the constrains of the formal structure have a number of (where $d = \sum_i n_i$)

$$R_N(n_1, \ldots, n_M) = \sum_{\lambda \vdash d \ \ell(\lambda) > N} \sum_{\mu} m^2_{\lambda\mu} \tag{12}$$

where $m_{\lambda\mu}$ means the coefficients when the representation λ of S_d is composed by the representation μ of $S_{n_1} \times \ldots S_{n_M}$.

Trace relations

A special case is for degree d of d different matrix, or the multidegree is $(1, \cdots, 1)$,

$$R_N(d) = \sum_{\ell(\lambda) > N} (\dim V_\lambda)^2 \tag{13}$$

another case is for degree d but only one matrix, or the multidegree is (d),

$$R_N(d) = p(d) - p_{\leq N}(d) \tag{14}$$

where $p(d)$ is the number of partitions of d, and $p_{\leq N}(d)$ counts when all parts $\leq N$. Interestingly, this is generated by

$$\sum_{d \geq 0} R_N(d) q^d = \prod_{k=1}^{\infty} \frac{1}{1 - q^k} - \prod_{k=1}^{N} \frac{1}{1 - q^k} \tag{15}$$

However, does all Fortuity comes from trace relations? Actually no, we can first see some simple examples where fortuity comes from simpler mechanisms.

Fortuity of $\mathcal{N}=2$ SYK

We first move on to a simple but intriguing model, which is the $\mathcal{N}=2$ SUSY SYK model. This model possess N complex fermions

$$\psi_i \quad i=1,\ldots,N \tag{16}$$

and a random supercharge with q odd

$$Q_N=\sum_{i_1<\cdots<i_q} C_{i_1\ldots i_q}\psi_{i_1}\ldots\psi_{i_q} \tag{17}$$

the Hilbert space with p fermions is clear as

$$\mathcal{H}_N^p \simeq \Lambda^p(\mathbb{C}^N) \quad \dim\mathcal{H}_N^p=\binom{N}{p} \tag{18}$$

we can view the Q action as a wedge product and Hilbert space as a space of forms

$$Q\alpha=C\wedge\alpha \quad \Lambda^p\mathbb{C}^N \overset{C\wedge}{\to} \Lambda^{p+q}\mathbb{C}^N \tag{19}$$

Fortuity of $\mathcal{N} = 2$ SYK

People discovered [8] that the BPS state, or Q cohomology, is strictly constrained in the SYK model

$$H^p(Q_N) = 0 \quad p < \frac{N-q}{2} \; or \; p > \frac{N+q}{2} \tag{20}$$

this means all BPS states are inside of the window $|2p - N| \leq q$, which is called the **Fortuity Window**, this phenomenon is called the **R charge concentration**.People also discovered that $\#_{fort} \sim e^{cN}$.

This means all BPS are actually Fortuity as well, because for every p, the state only remains BPS at $N \leq 2p + q$, we can find a N that the state is not BPS for every state. This makes SYK is a pure fortuitous laboratory without the effect of monotone states. However, since there's no role of trace relation here, the fortuitous and BPS only comes from the competing of ker Q_p and $\mathrm{im} Q_{p-q}$.

Fortuity of HTZ type of $\mathcal{N}=2$ SYK

Heydeman-Turiaci-Zhao [13] has extended the original $\mathcal{N}=2$ SYK model to a model that allows monotone states to appear. They introduce two types of fermions ψ_i, χ_i and

$$Q = \sum_{i_1<\cdots<i_q} C_{i_1\dots i_q}\psi_{i_1}\cdots\psi_{i_{q-1}}\bar{\chi}_{i_q} \tag{21}$$

For example $q=3$ we have $Q=\sum C_{ijk}\psi_i\psi_{(j}\bar{\chi}_{k)}$,this makes $V=\sum_i \psi_i\chi_i$ satisfies $[Q,V]=0$, then $V^n|\Omega\rangle$ is a monotone BPS state.

Fortuity of $\mathcal{N} = 2$ SYK

There's another interesting phenomenon named **BPS chaos**, which is describing the distribution of certain operator eigenvalues. They study the operator O by $\hat{O} = P_{BPS} O P_{BPS}$, where $P_{BPS} = \lim_{\beta \to \infty} e^{-\beta H}$ is a projection to ground-state subspace.

By analyzing the spectrum of $\hat{O}$ on monotone and fortuitous states of HTZ model,[8] find out that monotone BPS sector is highly more structured than fortuitous BPS sector, where the fortuitous BPS sector looks like a random subspace. The information entropy gives

$$S^{mono} \approx 2.70 \quad S^{fort} \approx 6.97 \tag{22}$$

Fortuity of D1-D5 CFT

A D1-D5 CFT is such a configuration: we have a TypeIIB string theory on $\mathbb{R}^1_t \times \mathbb{R}^4 \times S^1 \times M_4$, where M_4 is T^4 or K3. We let N_1 D1 branes wrapped on S^1 and N_5 D5 branes wrapped on $S^1 \times M_4$, we have $N = N_1 N_5$. The $\mathbb{R}^1_t \times S^1$ gives a $2\mathrm{d}$ SCFT with $\mathcal{N} = (4,4)$ supersymmetry. This theory enjoys a moduli orbifold point $\mathrm{Sym}^N(M_4) = \frac{(M_4)^N}{S_N}$ where the theory is greatly simplified.

States are organized by the twisted sectors of S_N and decomposed intro cycles

$$(1)_1^N(2)^{N_2}(3)^{N_3}\cdots \quad \sum_{w\geq 1} wN_w = N \tag{23}$$

A cycle of length w is called a strand (or long string), and can carry oscillator excitations.Thus a generic state has the schematic form

$$|\Psi\rangle = |\psi_{w_1}\rangle\,|\psi_{w_2}\rangle\cdots \quad \sum_i w_i = N \tag{24}$$

Unlike gauge theories, where finite N effects are encoded by trace relations, here finite N effects are encoded directly by the allowed total strand length.

Fortuity of D1-D5 CFT

It is useful to introduce an $N = \infty$ covering space containing arbitrary collections of nontrivial strands. However, at finite N, physical states must satisfy the **Stringy Exclusion Principle (SEP)**:

$$L_{nontrivial} = \sum_{nontrivial\ strands} w_i \leq N \tag{25}$$

The remaining length can be filled by vacuum $w = 1$ strands.

Fortuity of D1-D5 CFT

Away from the orbifold point, the exactly marginal deformation contains a twist-two operator. Therefore the relevant supercharge acts by joining and splitting strands:

$$(w_1) + (w_2) \overset{Q}{\leftrightarrow} (w_1 + w_2) \tag{26}$$

$$\begin{array}{ccc} |\Psi\rangle & \xrightarrow{Q} & |\Phi\rangle \\ L \leq N & & L > N \\ \text{physical} & & \text{removed by SEP} \end{array}$$

Hence a state that is not Q-closed in the covering theory may become BPS at finite N, because part of its Q-image is removed by the SEP. This is the origin of fortuity in D1-D5 CFT.

Fortuity of more complicated models

How do we compute fortuity for a more complicated theory? Especially a theory with more symmetries included, which we need to impose symmetries to all allowed operators.The standard process is already revealed by many studies.

First, we need to compute all possible BPS states. We need to choose a choice of nilpotent $Q, Q^\dagger$ at first, and compute their BPS condition.

$$\{Q, Q^\dagger\} = (BPS\ charge\ condition) \tag{27}$$

Fortuity of more complicated models

Secondly, we need to figure out all the fundamental components (as field components) we need to consider in the computing of fortuitous operators, these all called the **letters**. For a theory we can impose weak coupling and perturbative condition, we can simply choose all letters that are BPS as a free field (or **BPS letters**).

We need to compute how our choice of Q acts on different letters. This is simple since we can directly get the classical result by restrict the full SUSY transformation to our supercharge choice. We should be aware of the introducing of covariant derivative as a dressing of the letters, since it has a nontrivial commutation relation with Q action.

Then, for a specific charge sector q, we need to enumerate all possible single trace and multi trace operators. We denote all multi trace operators forming a space V_q.

Fortuity of more complicated models

We act Q on V_q to get QV_q, also we do the same enumeration on charge sector $q - q_0$, obtaining the QV_{q-q_0} space. This means we can get

$$\dim H_Q^q = \dim \ker Q - \dim \mathrm{Im} Q = \dim V_q - \dim(QV_q) - \dim(QV_{q-q_0}) \quad (28)$$

We then need to separate monotone states (multi gravitons) from all BPS states. We first examine all multi trace operators and find the subspace $W_q \subset V_q$, where W_q is defined by the subset of multi traces that all single traces are Q closed.

After this, we compute $W \cup QV_{q-q_0}$, the multi gravitons are

$$\dim \mathrm{span}(W_q \cup QV_{q-q_0}) - \dim(QV_{q-q_0}) \quad (29)$$

the number of fortuitous operators is the difference between number of all BPS states and monotone states.

Fortuity of more complicated models

One specific difficulty here is how to compute the dimension of the space of all multi trace operators. Fortunately, we can assume all operators in a adjoint representation of gauge symmetry to a different $N \times N$ matrix. We assume all its matrix elements, which is a $O(N^2)$ scale number. This enables us to expand all multi traces by these elements explicitly. By letting each **monomial as a basis vector**, we can compute the rank the matrix of all multi trace operators to know its dimension.

However, the step we mentioned above has some subtleties. What if we cannot use BPS letters directly? What if the Q action have quantum corrections?

One of the most important model is 4d $\mathcal{N} = 4$ SYM, which directly comes from the AdS/CFT of TypeIIB theory on $AdS_5 \times S^5$. The theory consists a $\mathcal{N} = 4$ vector multiplet, or 3 $\mathcal{N} = 1$ chiral and 1 $\mathcal{N} = 1$ vector.

The superconformal group of $\mathcal{N} = 4$ SYM is $PSU(2,2|4)$, its supercharges are $Q^I_\alpha, \bar{Q}_{I\dot{\alpha}}$. We choose $Q = Q^4_-$ and $Q^\dagger = S_4^-$, this gives the $\frac{1}{16}$-BPS sector by

$$2\{Q^4_-, S_4^-\} = E - J_1 - J_2 - R_1 - R_2 - R_3 \tag{30}$$

and the Q has a quantum number of

$$(E, J_1, J_2, R_1, R_2, R_3) = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \tag{31}$$

Fortuity of 4d $\mathcal{N}=4$ SYM

What are the BPS letters here? We know $\mathcal{N}=4$ SYM has $A_\mu, \Phi^{IJ}, \Psi_{I\alpha}, \bar{\Psi}^I_{\dot\alpha}$. The BPS letters are defined as following

$$\phi^i, \psi_i, \lambda_{\dot\alpha}, f, D_{\dot\alpha} \tag{32}$$

where $\phi^i = \Phi^{4i}$, $i=1,2,3$, $\psi_i = -i\Psi_{i+}$, $i=1,2,3$, $\lambda_{\dot\alpha} = \bar{\Psi}^4_{\dot\alpha}$, $\dot\alpha = \dot{+}, \dot{-}$, $f = -iF_{++}$, $D_{\dot\alpha} = D_{+\dot\alpha}$, they all satisfies $E = J_1 + J_2 + R_1 + R_2 + R_3$. This is computed easily by the representation of each field. We can also obtain the Q action as

$$[Q, \phi^i] = 0 \quad \{Q, \psi_i\} = -i\epsilon_{ijk}[\phi^j, \phi^k] \quad \{Q, \lambda_{\dot\alpha}\} = 0 \quad [Q, f] = i[\phi^i, \psi_i] \tag{33}$$

and relations to simplify

$$[Q, D_{\dot\alpha}\xi\} = -i[\lambda_{\dot\alpha}, \xi\} + D_{\dot\alpha}[Q, \xi\}] \quad [D_{\dot\alpha}, D_{\dot\beta}] = \epsilon_{\dot\alpha\dot\beta} f \quad D_{\dot\alpha}\lambda^{\dot\alpha} = [\phi^i, \psi_i] \tag{34}$$

One particular interesting thing is we can store all BPS letters to a superfield (where $\phi^n(z) = \sum_{r=0}^{\infty} \frac{(z^{\dot{\alpha}} D_{\dot{\alpha}})^r}{r!} \phi^n$)

$$\Psi(z, \theta) = -i[\lambda(z) + 2\theta_i \phi^i(z) + \epsilon^{ijk} \theta_i \theta_j \psi_k(z) + 4\theta_1\theta_2\theta_3 f(z)] \tag{35}$$

with the action

$$\{Q, \Psi\} = \Psi^2 \tag{36}$$

One subsector of this full BPS letter is the **BMN sector**, where we only preserve

$$\phi^i, \psi_i, f \tag{37}$$

BMN sector has a physical implication of truncating the $\frac{1}{16}$ BPS theory on the lowest excitations on S^3. This simplifies the theory while still include many fortuitous states.

Fortity of 4d $\mathcal{N} = 4$ SYM

There are plenty many of works calculating the fortuity of this model. The first O_0 was discovered by [1] has $E = \frac{19}{2}$

$$O_0 = \epsilon_{p_1 p_2 p_3}(\phi^m \cdot \psi_{p_1})(\phi^n \cdot \psi_{p_2})(\psi_m \cdot (\psi_n \times \psi_{p_3})) \tag{38}$$

in $SU(2)$, later this was extend to a infinite tower by [2]

$$O_n = (f \cdot f)^n O_0 + n(f \cdot f)^{n-1} f \cdot \xi(\phi, \psi) + \frac{2n^2 + n}{3}(f \cdot f)^{n-1}\chi(\phi, \psi) \tag{39}$$

also $O_m O_n = 0$.However, all these lies in the BMN sector. Later, [3] find a fermionic fortuitous state in $SU(3)$ $E = \frac{21}{2}$, this hints us the energy level we can find fortuitous state increases when N increases, which holds true as far as we know.

However, although later works extend our knowledge to $SU(4), SU(5), SU(6)$ BMN sector, they are mainly computing the BMN non-graviton index instead of giving the explicit form of operators. People find out that fortuity in BMN sector shows a distribution similar to black hole entropy, this may means that BMN sector actually includes many of the fortuitous operators in the full theory.

It is also discovered, while checking the duality between $SO(2N+1)$ and $Sp(N)$ group[14], the mismatch of $\frac{1}{16}$ BPS index revealed a $SO(7)$ fortuity and also explicitly constructed. Which may gives a new perspective of calculation.

We now describe the **partial no-hair** behavior discovered in $\mathcal{N} = 4$ SYM[2]. For fortuitous $\mathcal{O}$ and monotone G, OG is always Q closed but sometimes Q exact, implying a zero divisor in Q cohomology ring, or 'no hair'.

For the tower of core operators O_n, with $E(O_n) = \frac{19}{2} + 4n$ and $J_1 = J_2 = \frac{5}{2} + 2n$,some of gravitons G are not allowed to be added, as $\phi^{(m} \cdot \phi^{n)}$ or $\phi^m \cdot \lambda_{\dot{\alpha}}$, their product with O_0 results a Q exact form. But some others are allowed to be added, as

$$G^m = \phi^m \cdot f + \frac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p \tag{40}$$

Indicating that there might exist some selections of 'hair' graviton by blackhole.

One thing that is quite important is we need to notice all our Q actions are classical (tree) level, the Q should generally admits a **quantum correction**

$$Q = Q_0 + g_{YM}^2 Q_1 + O(g_{YM}^4) \tag{41}$$

also, the operator should also have a correction (Q_1 maps BMN to the full $\frac{1}{16}$ sector, where should the we consider instead of just BMN)

$$(Q_0 + g^2 Q_1)(O_0 + g^2 O_1) = Q_0 O_0 + g^2 (Q_1 O_0 + Q_0 O_1) \tag{42}$$

so people checked $Q_1 O_0$ if it is Q_0 exact to see if O is lifted by quantum effect, also make sure that $Q_1 O_0 \neq 0$. For the O_n tower, all operators are lifted except O_0, together with some of the hairs. They also observed that $Q_1 O_n$ actually a classical fortuity state, this means although the spectrum maybe quite different at 1 loop, but the entropy may have a close number, with only 1.2% difference[15].

Fortuity of ABJM

We now move on to ABJM. ABJM is similar to $\mathcal{N}=4$ SYM, but there are also some big differences. In ABJM, there is a manifest $\mathcal{N}=6$ supersymmetry in Lagrangian, but a $\mathcal{N}=8$ enhanced IR supersymmetry at $k=1,2$ in operator level. Also, ABJM admits a fundamental nonperturbative object which is **monopole**, this greatly affects people's understanding of ABJM. Also, ABJM is a Chern-Simons matter theory, this differs it from other $3\mathrm{d}$ gauge theories.

The definition of ABJM is as follows, there are two $U(N)$ gauge groups together with four $\mathcal{N}=2$ chirals as bifundamental, each two pointing two each other. The two gauge groups each carried Chern-Simons term has opposite CS level $k,-k$.

Fortuity of ABJM

The choice of supercharge here is doing a complex composition of superalgebra, $Q = Q_1^- + iQ_2^-$, $Q^\dagger = S_1^- - iS_2^-$ with the BPS condition

$$\{Q, Q^\dagger\} = 4E - 4J - 2(2R_1 + R_2 + R_3) \tag{43}$$

We have BPS letters as follows, when we ignore monopole at all

$$\phi_a, \bar{\phi}_a, \psi^a, \bar{\psi}_a, D \tag{44}$$

and Q actions as

$$[Q, \phi_a] = [Q, \bar{\phi}_a] = 0 \quad \{Q, \psi^a\} = \epsilon^{bc}\phi_b\bar{\phi}^a\phi_c \quad \{Q, \bar{\psi}^a\} = \epsilon^{bc}\bar{\phi}_b\phi^a\bar{\phi}_c \tag{45}$$

Use the same strategy in $\mathcal{N} = 4$ SYM, we get the results of ABJM.

Fortuity of ABJM

It is quite a good news that fortuitous states arrives at a much lower sector in ABJM then in $\mathcal{N}=4$ SYM. In [10], people use the index result to obtain the $N=2$ fortuity in $E+J=3$ level, this is done by enumerating all single trace gauge invariant words to formulate the I_{sp}, we have $\mathrm{Tr}(\phi_a\bar{\phi}_b)\in(\mathbf{2},\mathbf{2})$, $\mathrm{Tr}(\phi_{(i}\bar{\phi}_{(p}\phi_{j)}\bar{\phi}_{q)})\in(\mathbf{3},\mathbf{3})$, $\mathrm{Tr}(\phi_a\psi^a),\mathrm{Tr}(\bar{\phi}_a\bar{\psi}^a)$, these gives

$$I_{sp}=\chi_2(u)\chi_2(v)x+\left[\chi_3(u)\chi_3(v)-\chi_3(u)-\chi_3(v)-1\right]x^2+\cdots \qquad (46)$$

the multi graviton index is constructed by

$$I_{multi}=\exp\left(\sum_{m=1}^{\infty}\frac{1}{m}I_{sp}\right) \qquad (47)$$

then by calculating $I_{N,k}-I_{N,k}^{multi}$, it is known that there is a $(\mathbf{2},\mathbf{2})$ in $E+J=3$ (x^3), which is explicitly constructed.

Fortuity of ABJM

These first discovered states are

$$\mathcal{O} = \epsilon^{ij}\mathrm{Tr}(\psi_i\phi_j\bar{\phi}_k\phi_l + \bar{\psi}_i\bar{\phi}_j\phi_k\bar{\phi}_k) - \frac{3}{4}\epsilon^{ij}\mathrm{Tr}(\bar{\phi}_k\phi_l)\mathrm{Tr}(\bar{\psi}_i\bar{\phi}_j + \psi_j\phi_i) \quad (48)$$

this operator is obtained in large k, where monopole contribution of index decouples, we can also do a similar calculation in $k = 2$, where it is find $\mathbf{10} = (\mathbf{3}, \mathbf{1}) + (\mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{3})$ fortuitous states, where the middle one is the known one, the other two are monopole fortuitous states.

In [11], a brute force searching of fortuitous state is done in perturbative sector, and some of $N = 2$, $N = 3$, $N = 4$ states are found and explicitly constructed. Where $N = 2$ $E + J = 3, 4, 5$, $N = 3$ $E + J = 4, 5$, $N = 4$ $E + J = 5$ founds 244 fortuitous operators in total.

Fortuity of ABJM

Another particular interesting [11] found out is the relation between ABJM and the BMN sector of $\mathcal{N}=4$ SYM. However, this relation is not easy to find, since all fields are adjoint in $\mathcal{N}=4$ SYM, but only fundamental in ABJM. Therefore they correspond ϕ_{SYM} to $\phi\bar{\phi}$ since both have $E=1$, similarly, ψ_{SYM} to $\phi\bar{\psi}, \psi\bar{\phi}$ and f to $\psi\bar{\psi}, \phi D\bar{\phi}, D\phi\bar{\phi}$.

The **exact correspondence** they found is as follows

$$\phi^n_{SYM} \to X^n = \frac{1}{2}\phi_a\bar{\phi}^b(\sigma^n)^a_n \tag{49}$$

$$\psi^{SYM}_n \to \Psi_n = -\frac{1}{2}\left[\phi_a\bar{\psi}_b(\epsilon\sigma_n)^{ab} + \psi^a\bar{\phi}^b(\sigma_n\epsilon)_{ab}\right] \tag{50}$$

$$f_{SYM} \to F = \frac{i}{4}\left(2\psi^a\bar{\psi}_a - \phi_a D\bar{\phi}^a + D\phi_a\bar{\phi}^a\right) \tag{51}$$

which translates every BMN operator to a ABJM operator as a chain map $Q_{ABJM}\mathcal{F} = \mathcal{F}Q_{BMN}$. However, a subtlety here is that it is not sure that all these operators are not Q exact.

Fortuity of ABJM

Moreover, Behan also discovered that ABJM can be **composed to a superfield** satisfying the same superfield Q action as $\mathcal{N}=4$ SYM,

$$\begin{aligned}\mathcal{U}(z,\theta) = & -i\lambda\bar{\lambda}(z) - i\theta_n\,\phi_a\bar{\phi}^b(z)(\sigma^n)^a{}_b \\ & + \frac{i}{2}\epsilon_{mnp}\theta_m\theta_n\Big[\phi_a\bar{\psi}_b(z)(\epsilon\sigma_p)^{ab} + \psi^a\bar{\phi}^b(z)(\sigma_p\epsilon)_{ab}\Big] \\ & + \theta_1\theta_2\theta_3\Big[2\psi^a\bar{\psi}_a(z) - \phi_a D\bar{\phi}^a(z) + D\phi_a\bar{\phi}^a(z)\Big].\end{aligned} \tag{52}$$

satisfying $Q\mathcal{U}=\mathcal{U}^2$. This reveals a very useful way to study ABJM fortuity in $m=0$ sector from $\mathcal{N}=4$ SYM.

Another work recently [12] reveals more interesting aspects of ABJM theory by studying its **operator trajectory**. This is studied by obtaining the Hamiltonian using Gram matrix $G := \langle w_i | w_j \rangle$ (in this way, $Q^\dagger_{phys} = Q^\dagger_{matrix}$),

$$H(N) = \frac{1}{4}\left[QG^{-1}Q^\dagger G + G^{-1}Q^\dagger GQ\right] \tag{53}$$

and the operator trajectory is defined as the eigenvalue's dependence of N

$$H(N)v_i(N) = \gamma_i(N)v_i(N) \tag{54}$$

this is useful to see when the operator becomes a fortuitous operator by see when the trajectory has a zero. For example, the $N = 2$ $E + J = 3$ operator has the trajectory of

$$\gamma(N) = \frac{7N^2 - 10 - \sqrt{N^4 + 76N^2 + 4}}{k^2} \tag{55}$$

Also, we can consider the **partial hair** behavior for ABJM theory[12]. For example the graviton $G^c_d = \mathrm{Tr}(\bar{\phi}^c \phi_d)$ dressing the $N = 2$ $E + J = 3$ fortuitous operator O^a_b we mentioned before, the dressing result can be decomposed

$$(\mathbf{2}, \mathbf{2}) \otimes (\mathbf{2}, \mathbf{2}) = (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{3}) \tag{56}$$

only $(\mathbf{1}, \mathbf{1}), (\mathbf{3}, \mathbf{3})$ gives a new BPS state as $\epsilon_{ac}\epsilon^{bd} O^a_b \mathrm{Tr}(\bar{\phi}^c \phi_d)$.

Fortuity of ABJM

What more can we do to understand the fortuity of ABJM while monopole plays an important role here?

Maybe one possible anwser is introducing the **IR duality of ADHM and ABJM**. Since ADHM has a self-mirror mapping, there's some possibility that we can get more understanding by comparing these two theories.

However, since ADHM is a Yang-Mills theory instead of a Chern-Simons theory, there may be barriers and difficulties. But overall, this is worth trying.

Thanks for listening!

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